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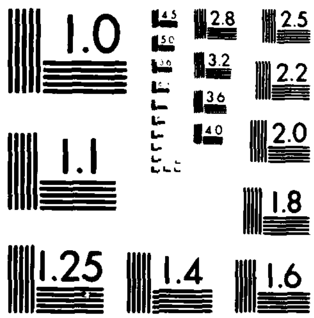

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L_∞-UPPER BOUND OF L_2-PROJECTIONS ONTO SPLINES AT A GEOMETRIC MESH

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ABSTRACT

For an integer $k > 1$ and a geometric mesh $(q^i)_{-\infty}^{\infty}$ with $q \in (0,\infty)$, let

$$M_{i,k}(x) := k[q^i,\ldots,q^{i+k}](\cdot-x)_+^{k-1}$$

$$N_{i,k}(x) := (q^{i+k} - q^i)M_{i,k}(x)/k \quad ,$$

and let $A_k(q)$ be the Gram matrix $(\int M_{i,k}N_{j,k})_{i,j\in\mathbb{Z}}$. It is known that $\|A_k(q)^{-1}\|_\infty$ is bounded independent of q. In this paper it is shown that $\|A_k(q)^{-1}\|_\infty$ is strictly decreasing for q in $[1,\infty)$. In particular, the sharp upper bound and lower bound for $A_k(q)^{-1}$ are obtained:

$$2k-1 < \|A_k(q)^{-1}\|_\infty < (\frac{\pi}{2})^{2k}\{\sum_{j\in\mathbb{Z}} (1+2j)^{-2k}\}^{-1} \quad \text{for all } q \in (0,\infty) \quad .$$

AMS (MOS) Subject Classification: 41A15

Key Words: splines, geometric mesh, least-squares approximation,
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SIGNIFICANCE AND EXPLANATION

Least-squares approximation by polynomial splines is a very effective means of approximation, particularly when the knot sequence can be chosen suitably nonuniform. The stability of this process can be linked to the norm of the inverse of the Gram matrix of a B-spline basis. For the special case when the knot sequence is geometric it is known that the norm of the inverse of that Gramian is bounded independently of the mesh ratio. Also, the sharp lower bound for the inverse of that Gramian is known.

In the present report, we continue these investigations and obtain, in particular, the sharp upper bound of the inverse of the Gram matrix.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

L_∞-UPPER BOUND OF L_2-PROJECTIONS ONTO SPLINES AT A GEOMETRIC MESH

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1. Introduction

Let $\underline{x} := (x_i)_{-\infty}^{\infty}$ be a strictly increasing biinfinite sequence with $x_{\pm\infty} := \lim_{i\to\pm\infty} x_i$ and $I := (x_{-\infty}, x_{+\infty})$. Let further

$$S := m\$_{k,\underline{x}}(I) := \{f \in C^{k-2}(I) \cap L_\infty(I) \;;\; f|_{(x_k,x_{i+1})} \text{ is a polynomial of degree } < k\}$$

be the normed linear space of bounded polynomial splines of order k with breakpoint sequence $\underline{x}$ and norm $\|f\| := \sup_{x\in I} |f(x)|$. We shall be concerned with P_S, the orthogonal projector onto S with respect to the ordinary inner product

$$(f,g) := \int_I f(x)g(x)dx \quad ,$$

but restricted to $L_\infty(I)$. We want to bound its norm

$$\|P_S\|_\infty := \sup_{f\in L_\infty(I)} \|P_S f\|_\infty / \|f\|_\infty \quad .$$

In 1973, de Boor raised the following

Conjecture [1]. $\qquad \sup_{\underline{x}} \|P_S\|_\infty < \text{const}_k < \infty$.

This conjecture has been verified for $k = 1,2,3,4$ (see de Boor [3] and the references cited there). de Boor [2] also obtained a bound of P_S in terms of a global mesh ratio. In general, however, this conjecture seems hard to solve. For geometric mesh $\underline{x}$, Höllig, K. [8] recently proved the boundedness of P_S. Later on, Feng, Y. Y. and Kozak, J. [6] reproved this result. Before recalling some results of theirs, we need to introduce some notations. For the mesh $\underline{x} = (x_i)_{-\infty}^{\infty}$, let

$$M_{i,k}(x) := k[x_i,\ldots,x_{i+k}](\cdot-x)_+^{k-1}$$

$$N_{i,k}(x) := ([x_{i+1},\ldots,x_{i+k}] - [x_i,\ldots,x_{i+k-1}])(\cdot-x)_+^{k-1}$$

$$= (x_{i+k} - x_i)M_{i,k}(x)/k \quad .$$

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Set

$$A_k(i,j) := \int M_{i,k} N_{j,k} \quad \text{for} \quad i,j \in \mathbb{Z} \ .$$

Let $A_k \in \mathbb{R}^{\mathbb{Z}\times\mathbb{Z}}$ be the biinfinite matrix given by the rule:

$$(i,j) \to A_k(i,j) \quad \text{for} \quad (i,j) \in \mathbb{Z}\times\mathbb{Z} \ .$$

It was shown by de Boor [1] that

$$D_k^{-2} \|A_k^{-1}\|_\infty \le \|P_S\|_\infty \le \|A_k^{-1}\|_\infty \ ,$$

where D_k is a constant depending only on k. Thus bounding P_S is equivalent to bounding A_k^{-1}.

Let us restrict ourselves now to a particular case where $\underline{x}$ is a geometric mesh: $\underline{x} := (q^i)_{-\infty}^{\infty}$ for some $q \in [1,\infty)$ (note that the case $q \in (0,1]$ is symmetric to the case $q \in [1,\infty)$; see [6] and [8]). Spline interpolation at a geometric mesh was first investigated by C. A. Micchelli [9], who based his argument on the properties of the so-called generalized Euler-Frobenius polynomials. Later on, Y. Y. Feng and J. Kozak [6] developed such a consideration. Earlier, and in a different way, K. Höllig [8] made a more precise investigation into the boundedness of L_2-projections onto splines on a geometric mesh. In particular, he got the following elegant result (see Theorem 5 in [8]):

Theorem A. For a geometric mesh $\underline{x} := (q^i)_{-\infty}^{\infty}$ with $q \in (0,\infty)$, let $A_k(q)$ be the biinfinite matrix $(\int M_{i,k} N_{j,k})_{i,j\in\mathbb{Z}}$. Then

$$\|A_k(q)^{-1}\|_\infty = |\Omega_k(q)|^{-1} \ , \tag{1}$$

where

$$\Omega_k(q) := 2\cdot k!\cdot(k-1)!\ t^{2k-1} \prod_{\nu=1}^{k} \frac{q^\nu+1}{q^\nu-1} \cdot \prod_{\nu=1}^{k-1} \frac{q^\nu+1}{q^\nu-1} \cdot \sum_{j\in\mathbb{Z}} \prod_{\nu=1}^{k} \frac{1}{[\pi(1+2j)]^2 + (\nu t)^2} \tag{2}$$

with $t := \log q$. Moreover,

$$\lim_{q\to1} \Omega_k(q) = \left(\frac{2}{\pi}\right)^{2k} \left\{ \sum_{j\in\mathbb{Z}} (1+2j)^{-2k} \right\} \tag{3}$$

$$\lim_{q\to\infty} \Omega_k(q) = \frac{1}{2k-1} \ . \tag{4}$$

Based on numerical evidence, de Boor raised the following

Conjecture. $\Omega_k(q)$ is a monotone increasing function on $[1,\infty)$.

This conjecture was verified for $k \leq q$ by Y. Y. Feng and J. Kozak [7]. They also showed that $\Omega_k(q) \leq \frac{1}{2k-1}$ in the same paper.

The purpose of this paper is to confirm the above conjecture. Thus we have

Theorem 1. $\Omega_k(q)$ is a monotone increasing function on $[1,\infty)$. In particular,

$$(5) \qquad 2k-1 \leq \|A_k(q)^{-1}\|_\infty \leq \left(\frac{\pi}{2}\right)^{2k}\left\{\sum_{j\in\mathbb{Z}} (1+2j)^{-2k}\right\}^{-1} .$$

Note that $\Omega_1(q) \equiv 1$ and that $\Omega_2(q) \equiv \frac{1}{3}$ in terms of a straightforward calculation. Hence we can restrict ourselves to the case $k \geq 3$ from now on.

In section 2, we will give an alternative proof of theorem A. Section 3 and 4 will be devoted to proving the monotonicity of $\Omega_k(q)$ for $q \in [1,20]$ and $q \in [20,\infty)$, respectively.

2. The bound for $A_k(q)^{-1}$.

As before, $\underline{x} = (q^i)_{-\infty}^{\infty}$ is a geometric mesh with $q \in (1,\infty)$ and $t = \log q$. Consider

$$\phi_0(x) := [0,1,\ldots,(2k-1)]_z \frac{x^z}{q^z+q^k} \quad \text{for} \quad x \in [1,q] \quad .$$

It is easy to verify that

$$(6) \qquad q^\ell\phi_0^{(\ell)}(q) + q^k\phi_0^{(\ell)}(1) = [0,1,\ldots,(2k-1)]_z\{z(z-1)\cdots(z-\ell+1)\} = \begin{cases} 0 & \text{for } \ell = 1,\ldots,2k-2 \\ 1 & \text{for } \ell = 2k-1 \end{cases} .$$

Since ϕ_0 is a polynomial of degree $2k-1$, $\phi_0^{(2k-1)}$ is constant in $[1,q]$. Hence (6) yields that

$$(7) \qquad \phi_0^{(2k-1)}(x) = \frac{1}{q^k+q^{2k-1}} \quad \text{for} \quad x \in [1,q] \quad .$$

Now we extend the domain of ϕ_0 to $(0,\infty)$ as follows:

$$\phi(x) := (-q^k)^m\phi_0(q^{-m}x) \quad \text{for} \quad q^m \leq x \leq q^{m+1}, \ m \in \mathbb{Z} \quad .$$

From (6) we assert that $\phi \in \mathcal{S}_{2k,\underline{x}}$, and that

$$(8) \qquad \phi(q^m) = (-q^k)^m\phi_0(1), \ m \in \mathbb{Z} \quad .$$

It follows that

$$[x_0,x_1,\ldots,x_{m-1},x_m]\phi = \frac{[x_1,\ldots,x_m]\phi - [x_0,\ldots,x_{m-1}]\phi}{x_m - x_0}$$

$$= \frac{-q^{k-m+1}[x_0,\ldots,x_{m-1}]\phi - [x_0,\ldots,x_{m-1}]\phi}{q^m - 1}$$

$$= -\frac{q^{k-m+1}+1}{q^m - 1}[x_0,\ldots,x_{m-1}]\phi \quad .$$

By induction on m, we can obtain

(9) $$[x_0,x_1,\ldots,x_k]\phi = (-1)^k\Big(\prod_{m=1}^{k} \frac{q^{k-m+1}+1}{q^m-1}\Big)\phi_0(1) = (-1)^k\cdot\Big(\prod_{m=1}^{k}\frac{q^m+1}{q^m-1}\Big)\cdot\phi_0(1) \quad .$$

From (8) we deduce that

(10) $$[x_i,\ldots,x_{i+k}]\phi = (-1)^i[x_0,\ldots,x_k]\phi \quad .$$

By Peano's theorem (see [4])

$$[x_i,\ldots,x_{i+k}]\phi = \int M_{i,k}(x)\phi^{(k)}(x)/k!\,dx \quad .$$

Now we get

(11) $$\int M_{i,k}(x)\phi^{(k)}(x)/k!\,dx = (-1)^i(-1)^k\Big(\prod_{m=1}^{k}\frac{q^m+1}{q^m-1}\Big)\phi_0(1) \quad .$$

Obviously, $\phi^{(k)}/k! \in \mathcal{S}_{k,\underline{x}}$; hence $\phi^{(k)}/k!$ may be expanded in a B-spline series:

$$\phi^{(k)}/k! = \Sigma\,\alpha_j N_{j,k} \quad .$$

However, $\phi^{(k)}(qx) = -\phi^{(k)}(x)$. Thus

$$\Sigma\,\alpha_j N_{j,k}(x) = -\Sigma\,\alpha_j N_{j,k}(qx) = -\Sigma\,\alpha_j N_{j-1,k}(x) = -\Sigma\,\alpha_{j+1}N_{j,k}(x) \quad .$$

By the uniqueness of B-spline expansion we assert that

$$\alpha_{j+1} = -\alpha_j \quad , \quad j \in \mathbb{Z} \quad .$$

Thus we can write

(12) $$\phi^{(k)}/k! = C\Sigma(-1)^j N_{j,k} \quad ,$$

where C is a constant to be determined. Now (11) and (12) together give

(13) $$\sum_{j\in\mathbb{Z}}(-1)^j\int M_{i,k}(x)N_{j,k}(x)dx = (-1)^i\,C^{-1}\,(-1)^k\Big(\prod_{m=1}^{k}\frac{q^m+1}{q^m-1}\Big)\phi_0(1) \quad .$$

Let

(14) $$\Omega_k(q) := C^{-1}(-1)^k\cdot\Big(\prod_{m=1}^{k}\frac{q^m+1}{q^m-1}\Big)\phi_0(1) \quad .$$

Then (see de Boor, C., S. Friedland and A. Pinkus [5])

$$\|A_k(q)^{-1}\|_\infty = |\Omega_k(q)|^{-1} .$$

It remains to determine C. Differentiate (12) k-1 times:

$$\phi^{(2k-1)}/k! = C(\Sigma(-1)^j N_{j,k})^{(k-1)} .$$

One the one hand,

$$\phi^{(2k-1)}(x)/k! = \frac{1}{k!} \cdot \frac{1}{q^k + q^{2k-1}} \quad \text{for} \quad x \in (1,q) .$$

On the other hand (see[4]),

$$(\Sigma\ (-1)^j N_{j,k})^{(k-1)} = \frac{2(k-1)}{q^{k-1}-1} \cdot \frac{(q+1)(k-2)}{q^{k-2}-1} \cdot \ldots \cdot \frac{q^{k-2}+1}{q-1} \cdot \sum_{j\in\mathbb{Z}} (-1)^j (q^{k-1})^{-j} N_{j,1}$$

$$= 2 \cdot (k-1)! \cdot \frac{1}{q^{k-1}+1} \cdot \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \cdot \sum_{j\in\mathbb{Z}} (-1)^j (q^{k-1})^{-j} N_{j,1} .$$

Thus, for $x \in (1,q)$

$$(\Sigma\ (-1)^j N_{j,k})^{(k-1)}(x) = 2 \cdot (k-1)! \cdot \frac{1}{q^{k-1}+1} \cdot \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} .$$

From the above calculation we get

$$\begin{aligned} C^{-1} &= k! \cdot (q^k + q^{2k-1}) \cdot 2 \cdot (k-1)! \cdot \frac{1}{q^{k-1}+1} \cdot \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \\ &= 2 \cdot k! \cdot (k-1)! \cdot q^k \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} . \end{aligned} \tag{15}$$

Finally, (14) and (15) yield that

$$\begin{aligned} \Omega_k(q) &= (-1)^k \cdot 2 \cdot k! \cdot (k-1)! \prod_{m=1}^{k} \frac{q^m+1}{q^m-1} \cdot \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \cdot q^k \quad \phi_0(1) \\ &= (-1)^k \cdot 2 \cdot k!(k-1)! \prod_{m=1}^{k} \frac{q^m+1}{q^m-1} \cdot \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \cdot q^k[0,1,\ldots,2k-1] \frac{1}{q^\bullet+q^k} . \end{aligned} \tag{16}$$

We follow the procedure in [9] and use a well-known formula for the divided difference to get

$$(-1)^k q^k [0,1,\ldots,2k-1] \frac{1}{q^\bullet+q^k} = \frac{(-1)^k q^k}{2\pi i} [\int_{C_R} - \sum_{j=0}^{2k-1} \int_{C_{r_j}} \frac{dz}{\left[\prod_{m=0}^{2k-1} (z-m)\right] \cdot (e^{z\log q}+q^k)} \tag{17}$$

where C_R and C_{r_j} stand for positively oriented circles with centers at 0 and j and radius R and r_j where R sufficiently large and r_j sufficiently small, $j = 0,1,\ldots,2k-1$.

Making $R \to \infty$, $r_j \to 0$ $(j = 0,1,\ldots,2k-1)$ in (17) and using the residue theorem we get

$$(-1)^k q^k [0,1,\ldots,2k-1] \frac{1}{q^{\cdot}+q^k} = (-1)^k q^k \cdot (-1) \sum_{j\in\mathbb{Z}} \operatorname{Res}_{z=i(\pi+2\pi j)/t+k} \frac{1}{\prod_{\nu=0}^{2k-1} (z-\nu)\cdot(e^{z\log q}+q^k)}$$

$$= (-1)^k \cdot \sum_{j\in\mathbb{Z}} \frac{1}{t^{2k-1} \prod_{\nu=0}^{2k-1} [\frac{i(\pi+2\pi j)}{t} - \nu+k]} = \sum_{j\in\mathbb{Z}} \frac{t^{2k-1}}{\prod_{\nu=0}^{2k-1} [\pi+2\pi j-i(k-\nu)t]}$$

$$= t^{2k-1} \cdot \sum_{j\in\mathbb{Z}} \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi j)^2+(\nu t)^2} .$$

Thus (2) is proved by substituting the above equality into (16). Then it is straightforward to verify (3). As to (4), we have

$$\lim_{q\to\infty} \Omega_k(q) = (-1)^k \cdot 2 \cdot k! \cdot (k-1)!/(2k-1)! \cdot \lim_{q\to\infty} \sum_{\ell=0}^{2k-1} (-1)^{\ell+1} \binom{2k-1}{\ell} \frac{q^k}{q^\ell+q^k}$$

$$= 2 \cdot (-1)^k/\binom{2k-1}{k} \cdot [\sum_{\ell=0}^{k-1} (-1)^{\ell+1} \binom{2k-1}{k} + (-1)^{k+1} \cdot \frac{1}{2} \cdot \binom{2k-1}{k}]$$

$$= 2 \cdot (-1)^k/\binom{2k-1}{k} [\sum_{\ell=0}^{k-1} (-1)^{\ell+1} [\binom{2k-2}{\ell-1} + \binom{2k-2}{\ell}] + (-1)^{k+1} \cdot \frac{1}{2} \cdot \binom{2k-1}{k}]$$

$$= 2 \cdot (-1)^k/\binom{2k-1}{k} [(-1)^k \binom{2k-2}{k-1} + (-1)^{k+1} \cdot \frac{1}{2} \binom{2k-1}{k}] = \frac{2k}{2k-1} - 1 = \frac{1}{2k-1} .$$

This ends the proof of Theorem A.

3. The monotonicity of $\Omega_k(q)$ for $q \in [1,20]$.

Recall $t = \log q$. Let

$$f_k(t) := t^{2k-1} \cdot \prod_{\nu=1}^{k} \frac{e^{\nu t}+1}{e^{\nu t}-1} \cdot \prod_{\nu=1}^{k-1} \frac{e^{\nu t}+1}{e^{\nu t}-1} \cdot \sum_{j\in\mathbb{Z}} \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi j)^2+(\nu t)^2} .$$

Then $\Omega_k(e^t) = 2 \cdot k!(k-1)! f_k(t)$. Consider $f_k'(t)/f_k(t)$. We have

$$(18) \qquad \frac{f_k'(t)}{f_k(t)} = \sum_{j=0}^{\infty} \frac{1}{t} u_{k,j}(t) f_{k,j}(t) \quad ,$$

where

$$u_{k,j}(t) := \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi j)^2+(\nu t)^2} / \{ \sum_{j=0}^{\infty} \prod_{\nu=1}^{k} \frac{1}{(\pi+2\pi j)^2+(\nu t)^2} \}$$

$$(19) \qquad f_{k,j}(t) := 2k-1 + \sum_{\nu=1}^{k-1} + \sum_{\nu=1}^{k} \left(\frac{\nu t e^{\nu t}}{e^{\nu t}+1} - \frac{\nu t e^{\nu t}}{e^{\nu t}-1}\right) - \sum_{\nu=1}^{k} \frac{2(\nu t)^2}{(\pi+2\pi j)^2+(\nu t)^2} \quad .$$

If we can show that $f_k'(t)/f_k(t) > 0$ for $t \in [0,3]$, then $\Omega_k'(q) > 0$ for $q \in [1,20]$, because $e^3 > 20$. For this it suffices to show $f_{k,0}(t) > 0$, since $f_{k,j}(t) > f_{k,0}(t)$ $(j = 1,2,\ldots)$ from (19). Let us first make the following observation.

Proposition 1.

$$\frac{\pi^2}{\pi^2+(cx)^2} > \frac{2xe^x}{e^{2x}-1} \quad \text{for} \quad x \in [0,\infty) \quad \text{and} \quad c \in [1,5/4] \quad .$$

Proof. Each of the following inequalities is equivalent to proposition 1:

$$e^{2x} - 1 > (1 + c^2x^2/\pi^2)2xe^x \quad ,$$

$$\sum_{n=0}^{\infty} 2^n x^n/(n+1)! > (1 + c^2x^2/\pi^2)(\sum_{n=0}^{\infty} x^n/n!) \quad ,$$

$$\sum_{n=2}^{\infty} 2^n x^n/(n+1)! > \sum_{n=2}^{\infty} [\frac{1}{n!} + \frac{c^2}{\pi^2} \cdot \frac{1}{(n-2)!}]x^n \quad .$$

However, an induction argument on n shows that

$$2^n/(n+1)! > \frac{1}{n!} + \frac{c^2}{\pi^2} \cdot \frac{1}{(n-2)!} \quad \text{for} \quad n > 2 \quad \text{and} \quad c \in [1,5/4] \quad .$$

Therefore proposition 1 is true.

Proposition 2.

$$\frac{2\pi^2}{\pi^2+(4x/3)^2} > \frac{\pi^2}{\pi^2+(5x/4)^2} + \frac{\pi^2}{\pi^2+(5x/3)^2} \quad .$$

Proof.

$$2(\pi^2 + 25x^2/9)(\pi^2 + 25x^2/16) = 2\pi^4 + 1250\pi^2x^2/144 + 625x^2/72$$

$$> 2\pi^4 + 1137\pi^2x^2/144 + 625x^2/81 = (\pi^2 + 16x^2/9)[(\pi^2 + 25x^2/9) + (\pi^2 + 25x^2/16)] \quad .$$

Multiplying the above inequality by $\pi^2/[(\pi^2 + \frac{16}{9}x^2)(\pi^2 + \frac{25}{16}x^2)(\pi^2 + \frac{25}{9}x^2)]$, we obtain proposition 2.

Proposition 3. $f_{k+1,0}(t) > f_{k,0}(t)$ for $t > 0$ and $k \geq 3$.

Proof. We shall argue by induction on k. For $k = 3$, we have

$$f_{4,0}(t) - f_{3,0}(t) = \frac{2\pi^2}{\pi^2+(4t)^2} - \frac{2 \cdot 3t \cdot e^{3t}}{e^{2 \cdot 3t}-1} - \frac{2 \cdot 4t \cdot e^{4t}}{e^{2 \cdot 4t}-1} .$$

Set $x := 3t$. Then proposition 1 and 2 yield that

$$f_{4,0}(t) - f_{3,0}(t) > \left(\frac{\pi^2}{\pi^2 + (5x/4)^2} - \frac{2xe^x}{e^{2x}-1}\right) + \left(\frac{\pi^2}{\pi^2+(5x/3)^2} - \frac{2 \cdot \frac{4}{3} xe^{4x/3}}{e^{2 \cdot 4x/3}-1}\right) > 0$$

Suppose now that $k \geq 4$. Then $\frac{k+1}{k} \leq \frac{5}{4}$. We have

$$f_{k+1,0}(t) - f_{k,0}(t) = \frac{2\pi^2}{\pi^2+[(k+1)t]^2} - \frac{2 \cdot kt \cdot e^{kt}}{e^{2kt}-1} - \frac{2(k+1)te^{(k+1)t}}{e^{2(k+1)t}-1}$$

$$= \left(\frac{\pi^2}{\pi^2+[(k+1)t]^2} - \frac{2 \cdot kt \cdot e^{kt}}{e^{2kt}-1}\right) + \left(\frac{\pi^2}{\pi^2+[(k+1)t]^2} - \frac{2 \cdot (k+1)t \cdot e^{(k+1)t}}{e^{2 \cdot (k+1)t}-1}\right) > 0 ,$$

according to proposition 1. Thus proposition 3 is proved.

Consequently, $f_{k,j}(t) > f_{3,0}(t)$ for all $k > 3$ and $j \geq 0$. The remaining task of this section is to elaborate the nonnegativity of $f_{3,0}(t)$. For this we need some estimates.

Proposition 4. Let $h(x) = \left(\frac{xe^x}{e^x+1} - \frac{1}{2}x\right) \cdot \frac{1}{x^2}$. Then $h'(x) < 0$ for $x > 0$.

Proof. $h(x) = \frac{e^x-1}{2x(e^x+1)}$ and

$$h'(x) = \frac{1}{2} \cdot \frac{x(e^x+1)e^x-(e^x-1)[(e^x+1)+xe^x]}{[x(e^x+1)]^2} = \frac{1+2xe^x-e^{2x}}{2x^2(e^x+1)^2} ,$$

while

$$1 + 2xe^x - e^{2x} = 1 + 2x \cdot \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = - \sum_{n=3}^{\infty} \frac{2(2^{n-1}-n)}{n!} x^n < 0, \text{ for } x > 0 .$$

Proposition 5. $-\frac{xe^x}{e^x-1} > -1 - \frac{1}{2}x - \frac{1}{12}x^2$

Proof. $(1 + \frac{1}{2}x + \frac{1}{12}x^2)(e^x-1) - xe^x = \sum_{n=2}^{\infty} \frac{x^{n+1}}{n!} - \sum_{n=1}^{\infty} \frac{1}{2} \cdot \frac{x^{n+1}}{n!} + \sum_{n=1}^{\infty} \frac{1}{12} \cdot \frac{x^{n+2}}{n!}$

$$= \sum_{n=5}^{\infty} \frac{1}{n!} \cdot \frac{n^2-7n+12}{12} x^n > 0 \quad \text{for} \quad x > 0 \ .$$

Now we are in a position to prove that $f_{3,0}(t) > 0$ for $t \in [0,0.3]$. Write down

$$f_{3,0}(t) = 2\left(1 + \frac{te^t}{e^t+1} - \frac{te^t}{e^t-1} - \frac{t^2}{t^2+\pi^2}\right) + 2\left(\frac{2te^{2t}}{e^{2t}+1} - \frac{2te^{2t}}{e^{2t}-1} - \frac{(2t)^2}{(2t)^2+\pi^2}\right)$$

$$+ \left(1 + \frac{3te^{3t}}{e^{3t}+1} - \frac{3te^{3t}}{e^{3t}-1} - \frac{(3t)^2}{(3t)^2+\pi^2}\right) - \frac{9t^2}{\pi^2+9t^2} \ .$$

It follows from proposition 4 that, for $t \in [0,0.3]$,

$$te^t/(e^t+1) - t/2 > h(0.3)\cdot t^2 > 0.248t^2$$

$$2te^{2t}/(e^{2t}+1) - 2t/2 > h(0.6)\cdot(2t)^2 > 0.242\cdot(2t)^2$$

$$3te^{3t}/(e^{3t}+1) - 3t/2 > h(0.9)\cdot(3t)^2 > 0.234\cdot(3t)^2 \ .$$

In connection with proposition 5, we obtain

$$2\left(1 + \frac{te^t}{e^t+1} - \frac{te^t}{e^t-1} - \frac{t^2}{t^2+\pi^2}\right) > 2\left(0.248 - \frac{1}{12} - \frac{1}{\pi^2}\right)t^2 > 0.126t^2 \ ,$$

$$2\left(1 + \frac{2te^{2t}}{e^{2t}+1} - \frac{2te^{2t}}{e^{2t}-1} - \frac{(2t)^2}{(2t)^2+\pi^2}\right) > 2\left(0.242 - \frac{1}{12} - \frac{1}{\pi^2}\right)(2t)^2 > 0.458t^2 \ ,$$

$$1 + \frac{3te^{3t}}{e^{3t}+1} - \frac{3te^{3t}}{e^{3t}-1} - \frac{(3t)^2}{(2t)^2+\pi^2} > \left(0.234 - \frac{1}{12} - \frac{1}{\pi^2}\right)(3t)^2 > 0.444t^2 \ , \ .$$

$$-\frac{9t^2}{\pi^2+9t^2} > -\frac{9t^2}{\pi^2} > -0.912t^2 \ .$$

As a conclusion,

$$f_{3,0}(t) > (0.126 + 0.458 + 0.444 - 0.912)t^2 = 0.116t^2 \ .$$

This shows that

$$f_k'(t) > 0 \quad \text{for} \quad t \in [0,0.3] \ .$$

The next case we are going to treat is that $t \in [0.3,3]$. Let

$$v(t) := \frac{2\pi^2}{t^2+\pi^2} + \frac{2\pi^2}{4t^2+\pi^2} + \frac{\pi^2-9t^2}{\pi^2+9t^2}$$

$$w(t) := \frac{4te^t}{e^{2t}-1} + \frac{8te^{2t}}{e^{4t}-1} + \frac{6te^{3t}}{e^{6t}-1}$$

Then $f_{3,0}(t) = v(t) - w(t)$. It is easily seen that $v'(t) \le 0$ for $t \in [0,\infty)$. We claim that $w'(t) \le 0$ $(0 \le t < \infty)$, too. This is guaranteed by the following proposition.

Proposition 6. Let $g(x) := \frac{2xe^x}{e^{2x}-1}$. Then $g'(x) \le 0$ for $x \ge 0$.

Proof. $g'(x) = -\frac{2e^x}{(e^{2x}-1)^2}(1 + x + xe^{2x} - e^{2x})$, while

$$1 + x + xe^{2x} - e^{2x} = (1+x) - (1-x)\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{n=3}^{\infty} \frac{(n-2)\cdot 2^{n-1}}{n!} x^n \ge 0 \text{ for } x \ge 0 .$$

Accordingly,

(20) $\quad f_{3,0}(t) = v(t) - w(t) \ge v(b) - w(a)$ for $t \in [a,b]$ with $0 < a < b$.

To determine the positivity of $f_{3,0}$ I wrote a Fortran program and found that

$$v(\tfrac{n+1}{100}) - w(\tfrac{n}{100}) \ge 0.001 \quad \text{for} \quad n = 30,31,\ldots,299 .$$

Thus by (20) we assert that

$$f_{3,0}(t) > 0 \quad \text{for} \quad t \in [\tfrac{n}{100}, \tfrac{n+1}{100}],\ n = 30,31,\ldots,299 .$$

Therefore

$$f_{3,0}(t) > 0 \quad \text{for} \quad t \in \bigcup_{n=30}^{299} [\tfrac{n}{100}, \tfrac{n+1}{100}] = [0.3,3] .$$

So far we have shown that $\Omega_k'(q) \ge 0$ for $q \in [1,20]$.

4. The monotonicity of $\Omega_k(q)$ for $q \in [20,\infty)$.

Let

(21) $$f(q) := (-1)^k\cdot(2k-1)!q^k[0,1,\ldots,2k-1]\frac{1}{q^\cdot+q^k} .$$

Then

$$f(q) = (-1)^k q^\ell \cdot \sum_{\ell=0}^{2k-1} \binom{2k-1}{\ell}(-1)^{\ell+1}\frac{1}{q^\ell+q^k} = (-1)^k \sum_{\ell=0}^{2k-1}(-1)^{\ell+1}\binom{2k-1}{\ell}\frac{1}{1+q^{\ell-k}}$$

$$= \sum_{\ell=0}^{k-1}(-1)^{k+\ell+1}\binom{2k-1}{\ell}\frac{q^{k-\ell}}{1+q^{k-\ell}} + (-1)\binom{2k-1}{k}\cdot\frac{1}{2} + \sum_{\ell=k+1}^{2k-1}(-1)^{k+\ell+1}\binom{2k-1}{\ell}\frac{1}{1+q^{\ell-k}}$$

It follows that

$$(22)\qquad f'(q) = \sum_{\ell=0}^{k-1} (-1)^{k+\ell+1}\binom{2k-1}{\ell} \frac{(k-\ell)q^{k-\ell-1}}{(1+q^{k-\ell})^2} + \sum_{\ell=k+1}^{2k-1} (-1)^{k+\ell}\binom{2k-1}{\ell} \frac{(\ell-k)q^{\ell-k-1}}{(1+q^{\ell-k})^2}$$

$$= \sum_{\ell=1}^{k} (-1)^{\ell-1}\binom{2k-1}{k-\ell}\cdot\frac{\ell q^{\ell-1}}{(1+q^{\ell})^2} - \sum_{\ell=1}^{k-1} (-1)^{\ell-1}\binom{2k-1}{k+\ell}\cdot\frac{\ell q^{\ell-1}}{(1+q^{\ell})^2}$$

$$= \sum_{\ell=1}^{k-1} (-1)^{\ell-1}\left[\binom{2k-1}{k-\ell} - \binom{2k-1}{k+\ell}\right] \frac{\ell q^{\ell-1}}{(1+q^{\ell})^2} + (-1)^{k-1}\frac{kq^{k-1}}{(1+q^k)^2} .$$

Now we need the following propositions.

Proposition 7. $\left[\binom{2k-1}{k-\ell} - \binom{2k-1}{k+\ell}\right]\cdot \frac{\ell q^{\ell-1}}{(1+q^{\ell})^2}$ decreases as ℓ increases and $q > 6$.

Proof.

$$(23)\qquad \binom{2k-1}{k-\ell} - \binom{2k-1}{k+\ell} = \frac{(2k-1)!}{(k-\ell)!(k-1+\ell)!}\left(1 - \frac{k-\ell}{k+\ell}\right) = \frac{(2k-1)!}{(k-\ell)!(k+\ell)!}\cdot 2\ell .$$

We want to show

$$(24)\qquad \frac{(2k-1)!}{(k-\ell)!(k+\ell)!}\cdot 2\ell\cdot\frac{\ell q^{\ell-1}}{(1+q^{\ell})^2} > \frac{(2k-1)!}{(k-\ell-1)!(k+\ell+1)!}\cdot 2(\ell+1)\cdot\frac{(\ell+1)q^{\ell}}{(1+q^{\ell+1})^2} \text{ for } q > 6 .$$

It is easily seen that (24) is equivalent to

$$\frac{1}{q}\left(\frac{q^{\ell+1}+1}{q^{\ell}+1}\right)^2 > \frac{k-\ell}{k+\ell+1}\left(1+\frac{1}{\ell}\right)^2 .$$

However,

$$\frac{1}{q}\left(\frac{q^{\ell+1}+1}{q^{\ell}+1}\right)^2 = \frac{q^{2\ell+2}+2q^{\ell+1}+1}{q(q^{2\ell}+2q^{\ell}+1)} > q - 2q^{1-\ell} > q-2 ,$$

because

$$(q-2q^{1-\ell})q(q^{2\ell}+2q^{\ell}+1) = q^{2\ell+2} - q^2 - 2q^{2-\ell} < q^{2\ell+2} + 2q^{\ell+1} + 1 .$$

Meanwhile

$$\frac{k-\ell}{k+\ell+1}\left(1+\frac{1}{\ell}\right)^2 < 4 .$$

Therefore (24) holds for $q > 6$, and proposition 7 is proved.

Proposition 8. For $k > 2$ and $q > 6$,

$$(25)\qquad f'(q) > \binom{2k-1}{k-1}\cdot\frac{2}{k+1}\cdot\frac{1}{(1+q)^2} - \binom{2k-1}{k-2}\cdot\frac{4}{k+2}\cdot\frac{2q}{(1+q^2)^2} .$$

In particular, $f'(q) > 0$ and

$$(26)\qquad f(q) < \lim_{q\to\infty} f(q) = \binom{2k-1}{k-1}\cdot\frac{1}{2(2k-1)} .$$

Proof. Suppose first k is even, $k = 2m$. Then (22) and (23) yield that

$$f'(q) = \binom{2k-1}{k-1} \cdot \frac{2}{k+1} \cdot \frac{1}{(1+q)^2} - \binom{2k-1}{k-2} \cdot \frac{4}{k+2} \cdot \frac{2q}{(1+q^2)^2}$$

$$+ \sum_{j=2}^{m-1} \Big[\frac{(2k-1)!}{(k-2j+1)!(k+2j-2)!} \cdot \frac{2(2j-1)}{k+2j-1} \cdot \frac{(2j-1)q^{2j-2}}{(1+q^{2j-1})^2}$$

$$- \frac{(2k-1)!}{(k-2j)!(k+2j-1)!} \cdot \frac{2\cdot 2j}{k+2j} \cdot \frac{2jq^{2j-1}}{(1+q^{2j})^2}\Big]$$

$$+ \Big[(2k-2) \cdot \frac{(k-1)q^{k-2}}{(1+q^{k-1})^2} - \frac{kq^{k-1}}{(1+q^k)^2}\Big] .$$

By proposition 7 all the terms under the summation sign are positive. Moreover,

$$q^{k-2}/(1+q^{k-1})^2 > q^{k-1}/(1+q^k)^2 \quad \text{for} \quad q > 1 ,$$

and

$$(2k-2)(k-1) \cdot \frac{q^{k-2}}{(1+q^{k-1})^2} - \frac{kq^{k-1}}{(1+q^k)^2} > [(2k-2)(k-1) - k] \frac{q^{k-1}}{(1+q^k)^2} > 0 \text{ for } k > 2 .$$

For odd k, the proof is similar. Thus (25) holds. Furthermore

$$f'(q) > \frac{2}{(1+q)^2} \frac{(2k-1)!}{(k-2)!(k+2)!} \Big[\frac{k+2}{k-1} - \frac{4(1+1/q)^2}{q(1+1/q^2)^2}\Big] > \frac{2}{(1+q)^2} \frac{(2k-1)!}{(k-2)(k+2)!} [1-4\cdot(1+\tfrac{1}{6})^2] \geq 0$$

$$\text{for} \quad q > 6 ,$$

and

$$f(q) < \lim_{q\to\infty} f(q) = \lim_{q\to\infty}\Big[\binom{2k-1}{k-1} \cdot \frac{1}{2} \cdot \Omega_k(q)\Big] = \binom{2k-1}{k-1} \cdot \frac{1}{2(2k-1)} .$$

Proposition 9. Let $S(q) = 4 \sum_{\nu=1}^{k-1} \frac{\nu q^{\nu-1}}{q^{2\nu}-1} + 2 \frac{kq^{k-1}}{q^{2k}-1}$. Then

$$S(q) < \frac{4q^2}{(q^2-1)(q-1)^2} \quad \text{for} \quad q > 1.$$

Proof. We have

$$S(q) = 4 \cdot \sum_{\nu=1}^{k-1} \frac{q^{2\nu}}{q^{2\nu-1}} \nu q^{-(\nu+1)} + 2k\cdot q^{-(k+1)} \cdot \frac{q^{2k}}{q^{2k}-1} .$$

Note that

$$\frac{q^{2\nu}}{q^{2\nu}-1} < \frac{q^2}{q^2-1} \quad \text{for} \quad \nu > 1 \quad \text{and} \quad q > 1 .$$

Hence

$$S(q) < \frac{4q^2}{q^2-1} \cdot \sum_{\nu=1}^{\infty} \nu q^{-(\nu+1)} = \frac{4q^2}{q^2-1} \cdot \frac{d}{dq}\left(-\sum_{\nu=1}^{\infty} q^{-\nu}\right) = \frac{4q^2}{q^2-1} \cdot \frac{d}{dq}\left(\frac{-1}{1-q^{-1}}\right) = \frac{4q^2}{(q^2-1)(q-1)^2} .$$

Proposition 10. Let $g_k(q) = \frac{2k-1}{k+1}\left[1 - \frac{4(k-1)}{k+2} \cdot \frac{q}{1+q^2}\right]$. Then

$$g_{k+1}(q) > g_k(q) \quad (k = 1,2,3,\ldots; q > 12) .$$

Proof.

$$g_{k+1}(q) - g_k(q) = \frac{3}{(k+1)(k+2)(k+3)}\left[(k+3)-(3k-1) \cdot 4 \cdot \frac{q}{1+q^2}\right]$$

$$> \frac{3}{(k+1)(k+2)(k+3)}\left[(k+3)-(3k-1) \cdot 4 \cdot \frac{12}{1+12^2}\right] > 0 \quad \text{for} \quad q > 12 .$$

Now we are in a position to prove the monotonicity of $\Omega_k(q)$ for $q \in [20,\infty)$. From (16) and (21) we see that

$$\Omega_k(q) = 2 \cdot k!(k-1)!/(2k-1)! \cdot \prod_{m=1}^{k} \frac{q^m+1}{q^m-1} \cdot \prod_{m=1}^{k-1} \frac{q^m+1}{q^m-1} \cdot f(q) .$$

Hence

$$\frac{\Omega_k'(q)}{\Omega_k(q)} = -\left(4 \sum_{\nu=1}^{k-1} \frac{\nu q^{\nu-1}}{q^{2\nu}-1} + 2\,\frac{kq^{k-1}}{q^{2k}-1}\right) + \frac{f'(q)}{f(q)} = -S(q) + \frac{f'(q)}{f(q)} .$$

By proposition 9 we have

$$S(q) < \frac{1}{q^2} \cdot \frac{4q^4}{(q^2-1)(q-1)^2} < \frac{1}{q^2} \cdot \frac{4 \cdot 20^4}{(20^2-1)(20-1)^2} < \frac{4.45}{q^2} \quad \text{for} \quad q > 20 .$$

Moreover, proposition 8 and 10 tell us that

$$\frac{f'(q)}{f(q)} > \frac{1}{(1+q)^2} \cdot \frac{4(2k-1)}{k+1} \cdot \left[1 - \frac{4(k-1)}{k+2} \cdot \frac{q}{1+q^2}\right] = \frac{4}{(1+q)^2}\, g_k(q)$$

$$> \frac{4}{(1+q)^2}\, g_4(q) = \frac{4}{(1+q)^2} \cdot \frac{7}{5} \cdot \left(1 - \frac{2q}{1+q^2}\right) = \frac{1}{q^2} \cdot \frac{28}{5}\, \frac{(1-1/q)^2}{(1+1/q)^2} \cdot \frac{1}{1+1/q^2}$$

$$> \frac{1}{q^2} \cdot \frac{28}{5} \cdot \frac{(1-\frac{1}{20})^2}{(1+\frac{1}{20})^2} \cdot \frac{1}{1+1/20^2} > \frac{4.55}{q^2} \quad \text{for} \quad q > 20 \quad \text{and} \quad k > 4 .$$

Therefore

$$\frac{\Omega_k'(q)}{\Omega_k(q)} = \frac{f'(q)}{f(q)} - S(q) > \frac{4.55}{q^2} - \frac{4.45}{q^2} = \frac{0.1}{q^2} > 0 \quad \text{for} \quad k > 4 \quad \text{and} \quad q > 20 .$$

It remains to check the case $k = 3$. For this we shall make a straightforward computation:

$$\Omega_3(q) = 24\cdot\left(\frac{q+1}{q-1}\right)^2\cdot\left(\frac{q^2+1}{q^2-1}\right)^2 \frac{q^3+1}{q^3-1}\cdot q^3\cdot(-1)[0,1,2,3,4,5]\,\frac{1}{q^{\cdot}+q^3}$$

$$= \frac{24}{120}\left(\frac{q+1}{q-1}\right)^2\cdot\left(\frac{q^2+1}{q^2-1}\right)^2\cdot\frac{q^3+1}{q^3-1}\cdot q^3\left(\frac{1}{1+q^3} - 5\,\frac{1}{q+q^3} + 10\,\frac{1}{q^2+q^3} - 10\cdot\frac{1}{2q^3} + 5\,\frac{1}{q^3+q^4} - \frac{1}{q^3+q^5}\right)$$

$$= \frac{1}{5}\cdot\frac{q^2+1}{q^2+q+1}\ .$$

Thus

$$\Omega_3'(q) = \frac{1}{5}\cdot\frac{q^2-1}{(q^2+q+1)^2} > 0 \quad \text{for} \quad q > 1\ .$$

This completes the proof of theorem 1.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

For an integer $k \geq 1$ and a geometric mesh $(q^i)_{-\infty}^{\infty}$ with $q \in (0,\infty)$, let

$$M_{i,k}(x) := k[q^i,\dots,q^{i+k}](\cdot - x)_+^{k-1}$$

$$N_{i,k}(x) := (q^{i+k} - q^i)M_{i,k}(x)/k ,$$

and let $A_k(q)$ be the Gram matrix $(\int M_{i,k}N_{j,k})_{i,j\in\mathbb{Z}}$. It is known that

(continued)

ABSTRACT (continued)

$\|A_k(q)^{-1}\|_\infty$ is bounded independent of q. In this paper it is shown that $\|A_k(q)^{-1}\|_\infty$ is strictly decreasing for q in $[1,\infty)$. In particular, the sharp upper bound and lower bound for $A_k(q)^{-1}$ are obtained:

$$2k-1 \le \|A_k(q)^{-1}\|_\infty \le (\frac{\pi}{2})^{2k}\{\sum_{j\in\mathbb{Z}} (1+2j)^{-2k}\}^{-1} \quad \text{for all } q \in (0,\infty) .$$

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